Quantum Information Theory

Linear algebra review

Hermitian matrices: $M \in \mathbb{C}^{d \times d}$ s.t. $M = \overline{M^T} = M^*$

- Orthonormal basis of eigenvectors
- Real eigenvalues

$$M = \sum_{i=1}^{d} \lambda_i \cdot v_i v_i^* = \sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i|$$

Positive semidefinite matrices: $M \succcurlyeq 0$

- Hermitian with eigenvalues ≥ 0
- $\langle u, Mu \rangle = \langle u, \sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i| u \rangle = \sum \lambda_i |\langle u, v_i \rangle|^2 \geq 0$ $\forall u$

Trace: $\mathrm{Tr}(M) = \sum_i M_{ii} = \sum_i \lambda_i$, $\mathrm{Tr}(A^* B) = \langle A, B \rangle$

What is a quantum state?

$\psi \in \mathbb{C}^d \quad , \quad \|\psi\| = 1$

If we pick a basis: $|w_1\rangle , \ldots , |w_n\rangle$

$$|\psi\rangle = c_1 \cdot |w_1\rangle + \cdots + c_n \cdot |w_n\rangle \qquad \sum |c_i|^2 = 1$$

"measurement" $\longrightarrow |w_i\rangle$ w.p. $|c_i|^2 = \langle \psi , w_i \rangle^2$

Two kinds of uncertainties:

- State itself
- Outcome of measurement

Quantum Probability Theory

outcome $\longmapsto$ state

distribution $\longmapsto$ "mixed state"

"uncertain φ" : $\left.\begin{array}{l} \varphi_1 \text{ w.p. } p_1 \\ \vdots \\ \varphi_r \text{ w.p. } p_r \end{array}\right\} \in \mathbb{C}^d$

outcome in basis $w_1 \dots w_d$ = ?

$$P(\text{outcome "1"}) = \sum p_i \left|\langle \varphi_i, w_1 \rangle\right|^2$$

$$= \langle w_1, \sum_i p_i \varphi_i \varphi_i^* w_1 \rangle$$

$$= \langle \underbrace{\boxed{\sum_i p_i \varphi_i \varphi_i^*}}_{\rho}, w_1 w_1^* \rangle$$

$$\mathrm{Tr}(\rho) = \sum p_i \, \mathrm{Tr}(\varphi_i \varphi_i^*)$$

$$= \sum p_i \, \mathrm{Tr}(\varphi_i^* \varphi_i)$$

$$= 1$$

Distributions $\longrightarrow$ density matrice

$$\left.\begin{array}{lll} \varphi_1 & \text{w.p.} & p_1 \\ \vdots & & \\ \varphi_r & \text{w.p.} & p_r \end{array}\right\} \longrightarrow \rho = \sum p_i |\varphi_i\rangle\langle\varphi_i| \in \mathbb{C}^{d\times d}$$

$$\rho \geq 0 \quad , \quad Tr(\rho) = 1$$

Describe a mixed quantum state by density matrix ρ.

Measurement in $|w_1\rangle \dots |w_d\rangle$:

Outcome "i" w.p. $\langle w_i|\rho|w_i\rangle = \langle \rho, |w_i\rangle\langle w_i| \rangle$

Events

Measurement: $|w_1\rangle, \ldots\ldots, |w_d\rangle$

$\downarrow$

$\langle \rho, |w_1\rangle\langle w_1| \rangle \ldots\ldots \langle \rho, |w_d\rangle\langle w_d| \rangle$

$$|w_1\rangle\langle w_1| + \cdots\cdots + |w_d\rangle\langle w_d| = \mathbb{I}$$

POVM: $E_1 + \cdots + E_k = \mathbb{I}$ $\quad E_i \succcurlyeq 0$

outcome "i" w.p. $\langle \rho, E_i \rangle$

Ex: $\langle \rho, E_i \rangle \geq 0 \ \forall i$ $\quad \sum_i \langle \rho, E_i \rangle = 1$

Random Variables $\rightarrow$ "Observables"

Classical: $X : [d] \rightarrow \mathbb{R}$ or $X : [d] \rightarrow \mathbb{C}$ $\rightarrow$ $\begin{pmatrix} X(1) \\ \vdots \\ X(d) \end{pmatrix}$

Quantum: $X \in \mathbb{C}^{d \times d}$, Hermitian

$$X = \sum \underbrace{\lambda_i \, |v_i\rangle\langle v_i|}$$

$\hookrightarrow$ function with chosen basis

Measure in $|v_1\rangle \ldots \quad |v_d\rangle$

$\quad\quad\quad\quad\quad\quad\quad \downarrow \quad\quad\quad\quad \downarrow$

$\quad\quad\quad\quad\quad\quad\quad \lambda_1 \quad\quad\quad\quad \lambda_d.$

ρ

$$\text{Expected outcome} = \sum \lambda_i \langle \rho, |v_i\rangle\langle v_i| \rangle$$

$$= \langle \rho, \sum \lambda_i \, |v_i\rangle\langle v_i| \rangle$$

$$= \langle \rho, X \rangle$$

Expectations, variances, ...

For "observable" X

$$\mathbb{E}_\rho[X] = \langle \rho, X \rangle$$

$$\mathbb{E}_\rho[X^2] = \langle \rho, X^2 \rangle$$

$$\mathrm{Var}_\rho[X] = \mathbb{E}_\rho[X^2] - \left(\mathbb{E}_\rho[X]\right)^2$$

Non-commutativity

$XY \neq YX$, XY may not even be Hermitian von Neumann entropy

For density matrix $\rho \in \mathbb{C}^{d \times d}$

$$S(\rho) = \operatorname{Tr}\left(\rho \log \frac{1}{\rho}\right) = -\operatorname{Tr}\left(\rho \log \rho\right)$$

For any Hermitian $M \in \mathbb{C}^{d \times d}$,

$$M = \sum \lambda_i v_i v_i^* \qquad f(M) = \sum f(\lambda_i) \cdot v_i v_i^*$$

$$S(\rho) = \operatorname{Tr}\left(\sum_i \lambda_i \log \frac{1}{\lambda_i} \varphi_i \varphi_i^*\right) = \sum_i \lambda_i \log \frac{1}{\lambda_i}$$

Computing the entropy

- $\varphi = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ $\quad\quad \rho = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

  $S(\rho) = 1 \log 1 = 0$

- $\begin{cases} \varphi_1 = |0\rangle & \text{wp } 1/2 \\ \varphi_2 = |1\rangle & \text{wp } 1/2 \end{cases}$ $\quad\quad \rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

  $S(\rho) = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = 1$

<u>Ex:</u> $\rho = \sum \lambda_i \varphi_i \varphi_i^*$, measurement in basis $|w_1\rangle \dots |w_d\rangle$.

If outcome distribution is (classical) p, $H(p) = ?$

Prove $H(p)$ minimized for basis $|\varphi_1\rangle \dots |\varphi_d\rangle$.

[Hint: Jensen's]

Two parties: joint and marginal distributions

$$(\text{Alice} \quad , \quad \text{Bob})$$

$0/1$ $\quad$ $0/1$

$\varphi_A \in \mathbb{C}^2$ $\quad$ $\varphi_B \in \mathbb{C}^2$ $\quad$ $\varphi_{AB} \in \mathbb{C}^4$ (joint)

Jointly: ?

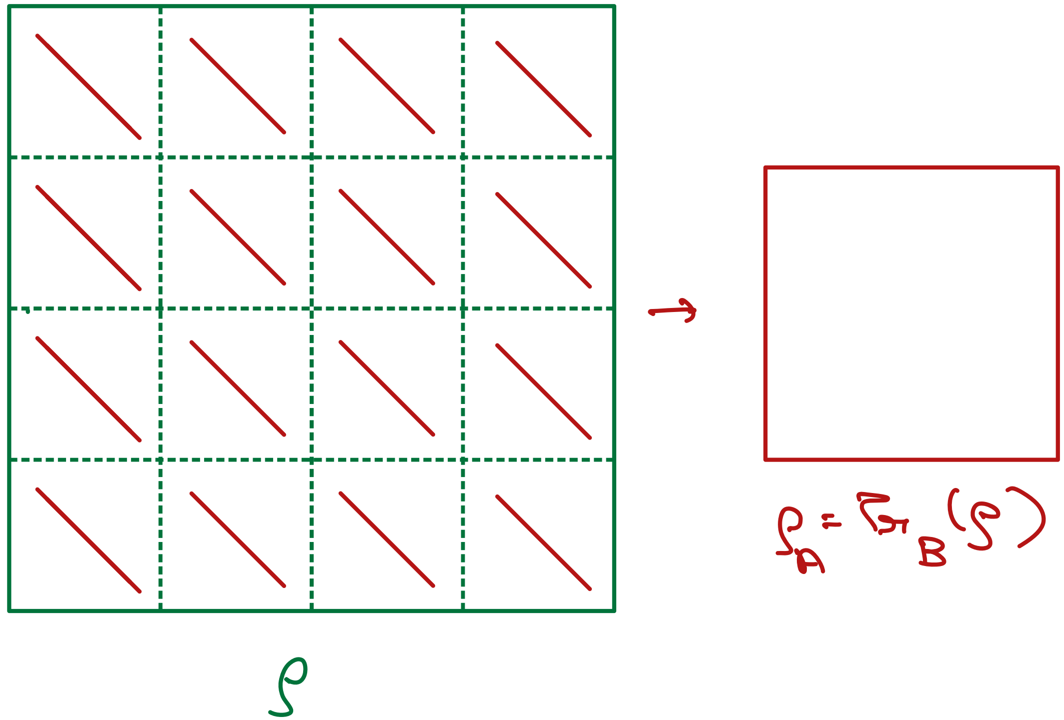
- "product" $\quad \rho_A \otimes \rho_B \in \mathbb{C}^{4\times 4}$
- "classical correlations": $\quad \sum_r \alpha_r \cdot \rho_{A,r} \otimes \rho_{B,r}$. $\sum \alpha_r = 1$
- "entangled": $\quad \rho \in \mathbb{C}^{4\times 4}$ (minus above cases)

e.g. $\quad \varphi = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$

Marginals

Given $\rho_{AB} : \mathbb{C}^{d_A \times d_B} \to \mathbb{C}^{d_A \times d_B}$

$\rho_A = ?$ $\rho_B = ?$



should satisfy

$\langle X \otimes I, \rho \rangle = \langle X, \rho_A \rangle$

for all observables X

$f(X) = \langle X \otimes I, \rho \rangle$ linear map on X.

$\exists$ unique ρ_A s.t. $\langle \rho_A, X \rangle = f(X)$

An example

$$\varphi = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \qquad \rho =$$

$$\rho_A = \qquad\qquad \rho_B =$$

$S(\rho) =$

$S(\rho_A) =$

$S(\rho_B) =$